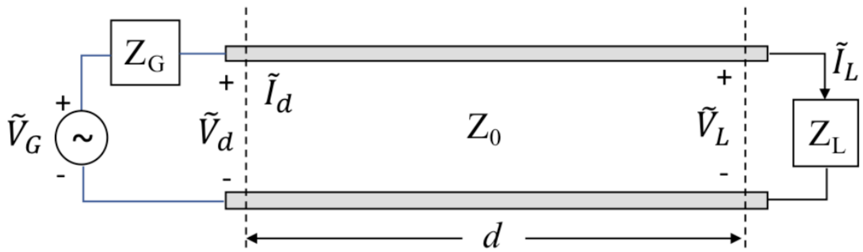
Problem 1: Voltage and Current along Transmission Line [+20 pts]

A lossless transmission line of characteristic impedance $Z_0 = 50\ \Omega$ and length $d = 17$ m is connected to an unknown load Z_L to a generator $\tilde{V}_G = 10\angle 0°$ volts as shown in the figure below. The internal impedance Z_g of the generator is unknown. The signal wavelength on the line is $\lambda = 8$ m. The current and voltage on the line at the generator end are measured and found to be $\tilde{I}_d = 40\angle 0°$ mA and $\tilde{V}_d = 6\angle 0°$ volts.



a **Determine the wave impedance Z_d at the generator end, as well as the generator's internal impedance Z_G.**

We can find the wave impedance, given by:

$$Z_d = \frac{\tilde{V}_d}{\tilde{I}_d} = \frac{6}{0.040} = 150\ \Omega$$

The wave impedance at $z = d$ is the input impedance of the transmission line. We can use voltage division to relate the generator voltage to the voltage at z = d, and rearrange to find the internal impedance of the generator.

$$\tilde{V}_d = \tilde{V}_g \frac{Z_{in}}{Z_g + Z_{in}}$$

$$Z_g = Z_{in} \frac{\tilde{V}_g - \tilde{V}_d}{\tilde{V}_d} = 150 \frac{10 - 6}{6} = 100\ \Omega$$

b **Determine the load impedance Z_L.**

We can rearrange the input impedance equation to solve for the load impedance:

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}$$

$$Z_L = Z_0 \frac{jZ_0 \tan(\beta l) - Z_{in}}{jZ_{in} \tan(\beta l) - Z_0}$$

where $\beta l = \frac{2\pi l}{\lambda}$, therefore

$$Z_L = 50\frac{j(50)\tan(\frac{2\pi(17)}{8}) - 150}{j(150)\tan(\frac{2\pi(17)}{8}) - 50} = 50\left[\frac{-150 + j(50)\tan(\frac{17\pi}{4})}{-50 + j(150)\tan(\frac{17\pi}{4})}\right] = 50\left[\frac{-150 + j(50)}{-50 + j(150)}\right]$$

$$= 50\left[\frac{\sqrt{(-150)^2 + (50)^2}e^{j\left(\arctan\frac{50}{-150}\right)}}{\sqrt{(-50)^2 + (-150)^2}e^{j\left(\arctan\frac{150}{-50}\right)}}\right] = 50e^{j(\arctan(-1/3) - \arctan(-3))}$$

$$= 50e^{j0.927} = 30 + j40\ \Omega$$

Problem 2: Standing Wave Ratio [+20 pts]

Consider a lossless transmission line with a characteristic impedance of 100Ω that is terminated with an unknown load impedance. The line is operated at a frequency corresponding to a wavelength $\lambda = 40$ cm. The standing wave ratio along this line is measured to be S = 3. The distance from the load to the first voltage minimum is measured to be 5 cm.

Based on these two measurements, determine the unknown load impedance.

We start with the equation relating the load impedance to the characteristic impedance and reflection coefficient:

$$Z_L = Z_0\left[\frac{1+\Gamma}{1-\Gamma}\right]$$

We can find Γ by separately calculating its magnitude and phase. Its magnitude is given by the standing wave ratio:

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5$$

and its phase θ_r is given by the following:

$$2\beta d_{min} - \theta_r = (2n+1)\pi$$

where n = 0 for our first voltage minimum. Additionally, $\beta = \frac{2\pi}{\lambda} = \frac{\pi}{20}$ rad/cm.
Therefore, we have

$$\theta_r = 2\beta d_{min} - \pi = 2\left(\frac{\pi}{20}\right)5 - \pi = -\frac{\pi}{2}$$

Now, we can combine these to find the reflection coefficient:

$$\Gamma = |\Gamma|e^{j\theta_r} = 0.5e^{-j\frac{\pi}{2}} = -j0.5$$

Resulting in

$$Z_L = 100\left[\frac{1-j0.5}{1+j0.5}\right] = 60 - j80\ \Omega$$

Problem 3: Power Flow in Lossless Transmission Line [+30 pts]

An antenna with a load impedance, $Z_L = (75 + j25)\Omega$, is connected to a transmitter through a 50Ω lossless transmission line. If under matched conditions, the transmitter can deliver 46W to the load, obtain the power that can be delivered to the antenna. Assume that $Z_g = Z_0$.

The net average power flowing towards (and then absorbed by) the load is

$$P_{av} = P^i_{av} + P^r_{av} = \frac{|V_0^+|^2}{2Z_0}\left(1 - |\Gamma|^2\right) \quad (W)$$

Where P^i_{av} is the time-average power of the incident wave and P^r_{av} is the time-average power of the reflected wave.

Recall that

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{1}$$

For a matched line, $Z_L = Z_0$, therefore $\Gamma = 0$, giving us

$$46(W) = \frac{|V_0^+|^2}{2Z_0} \Longrightarrow \sqrt{4600} = |V_0^+| \tag{2}$$

We can now solve for $\tilde{V}_g$, which remains unchanged when Z_L or Z_0 is changed

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}\right)\left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}}\right) \tag{3}$$

$$\tilde{V}_g = V_0^+ \frac{Z_g + Z_{in}}{Z_{in}}\left(e^{j\beta l} + \Gamma e^{-j\beta l}\right) \tag{4}$$

Knowing that

$$Z_{in} = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right) = Z_0 \left(\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} \right) \cdot \frac{e^{j\beta l}}{e^{j\beta l}} = Z_0 \left(\frac{e^{j\beta l}+\Gamma e^{-j\beta l}}{e^{j\beta l}-\Gamma e^{-j\beta l}} \right) \tag{5}$$

As determined before, for a matched line, $\Gamma = 0$, so

$$Z_{in} = Z_0 \tag{6}$$

So

$$\tilde{V}_g = V_0^+ \frac{Z_g + Z_{in}}{Z_{in}} \left(e^{j\beta l} + \Gamma e^{-j\beta l} \right) = V_0^+ \frac{Z_g + Z_0}{Z_0} \left(e^{j\beta l} \right) = |V_0^+| \frac{Z_g + Z_0}{Z_0} \left(e^{j(\beta l + \angle V_0^+)} \right) \tag{7}$$

But we know that for a matched line $\angle V_0^+ = 0$.

$$\tilde{V}_g = |V_0^+| \frac{Z_g + Z_0}{Z_0} \left(e^{j\beta l} \right) = \sqrt{4600} \frac{50+50}{50} \left(e^{j\beta l} \right) = 2\sqrt{4600} \left(e^{j\beta l} \right) \tag{8}$$

With our new $Z_L = (75 + j25)\Omega$, Γ becomes

$$\Gamma = \frac{75 + j25 - 50}{75 + j25 + 50} = \frac{25 + j25}{125 + j25} = \frac{35.35 e^{j0.79}}{127.48 e^{j0.197}} = 0.277 e^{j0.59} \tag{9}$$

$$|\Gamma| = 0.277 \Longrightarrow |\Gamma|^2 = 0.077 \tag{10}$$

And our new $|V_0^+|$ becomes

$$\begin{aligned}
|V_0^+| &= \left| \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right) \right| \\
&= \left| \left(\frac{2\sqrt{4600} \left(e^{j\beta l} \right) Z_0 \left(\frac{e^{j\beta l}+\Gamma e^{-j\beta l}}{e^{j\beta l}-\Gamma e^{-j\beta l}} \right)}{50 + Z_0 \left(\frac{e^{j\beta l}+\Gamma e^{-j\beta l}}{e^{j\beta l}-\Gamma e^{-j\beta l}} \right)} \right) \left(\frac{1}{e^{j\beta l} + \Gamma e^{-j\beta l}} \right) \right| \\
&= \left| \left(\frac{2\sqrt{4600} \left(e^{j\beta l} \right) 50 \left(\frac{1}{1} \right)}{50 \left(e^{j\beta l} - \Gamma e^{-j\beta l} \right) + 50 \left(e^{j\beta l} + \Gamma e^{-j\beta l} \right)} \right) \left(\frac{1}{1} \right) \right| \\
&= \left| \frac{2\sqrt{4600} \left(e^{j\beta l} \right) 50}{100 e^{j\beta l}} \right| \\
&= \sqrt{4600}
\end{aligned} \tag{11}$$

Note that when $Z_g = Z_0$, changing Z_L does not effect V_0^+. So our new delivered P_{av} becomes.

$$P_{av} = \frac{|V_0^+|^2}{2Z_0} \left(1 - |\Gamma|^2 \right) = 46 \cdot (1 - 0.077) = 42.47 \quad (W) \tag{12}$$

Problem 4: Transients on Transmission Lines [+30 pts]

We have a 1 m long lossless line characterized by $Z_0 = 50\Omega$ and $u_p = 2c/3$ (where c is the velocity of light). The line is fed by a step voltage applied at $t = 0$ by a generator circuit with $V_g = 60$ V and $R_g = 100\Omega$. The line is terminated in a load $R_L = 25\Omega$.

a **Generate a bounce diagram for the voltage $V(z,t)$ on this line.**

First find the reflection coefficient at the generator:

$$\Gamma_g = \frac{R_g - Z_0}{R_g + Z_0} = \frac{100 - 50}{100 + 50} = \frac{50}{150} = \frac{1}{3}$$

Then find the reflection coefficient at the load:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{25 - 50}{25 + 50} = \frac{-25}{75} = \frac{-1}{3}$$

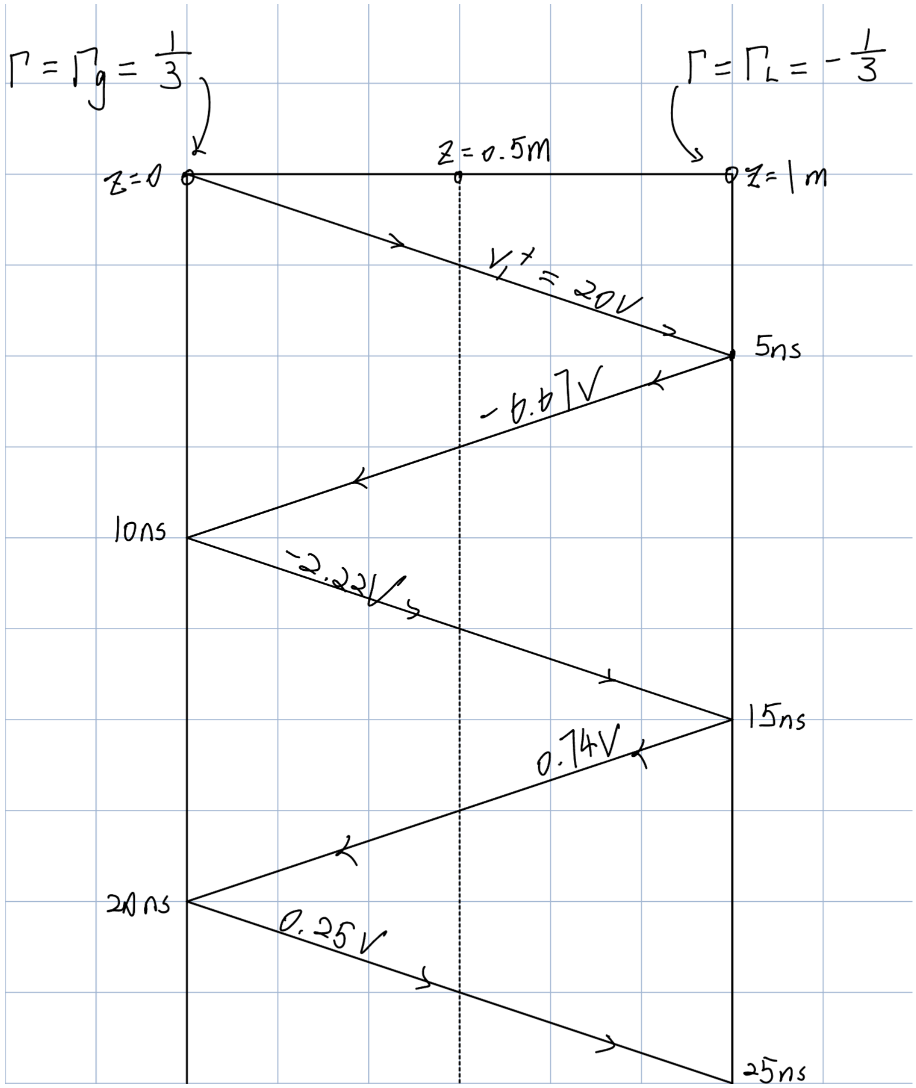
The initial voltage V_1^+ at the sending end of the transmission line is given by:

$$V_1^+ = \frac{V_g Z_0}{R_g + Z_0} = \frac{60 \times 50}{100 + 50} = 20 \text{ V}$$

And the time it takes the wave to travel the full length of the line is given by:

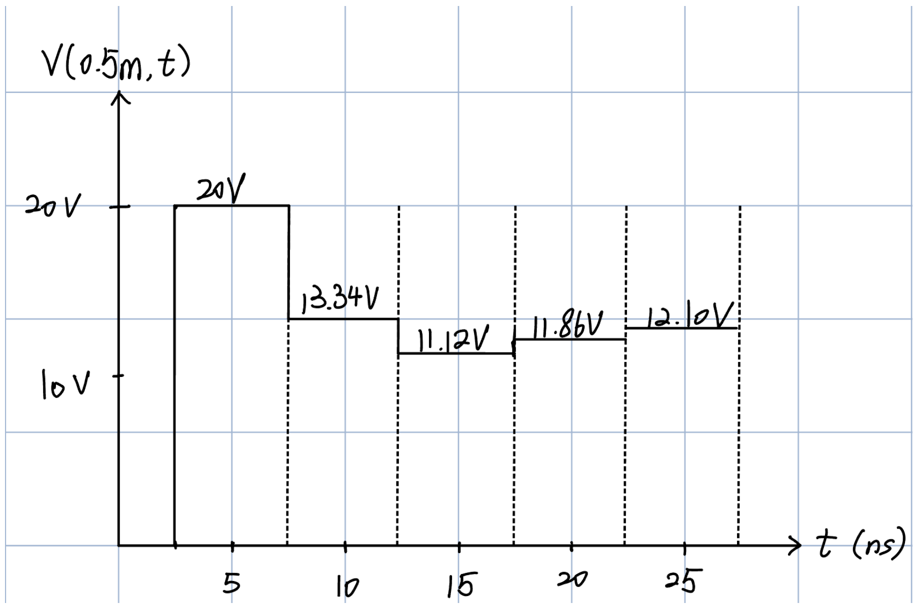
$$T = \frac{l}{u_p} = \frac{l}{2c/3} = \frac{1}{2 \times 10^8} = 5 \text{ ns}$$

From these information, we can generate the bounce diagram. At each bounce, we multiply the incident voltage wave by the respective reflection coefficient at each end.


$\Gamma = \Gamma_g = \frac{1}{3}$
$\Gamma = \Gamma_L = -\frac{1}{3}$
$z = 0$
$z = 0.5m$
$z = 1m$
$V_1^+ = 20V$
5ns
$-6.67V$
10ns
$-2.22V$
15ns
0.74V
20ns
0.25V
25ns

b **Use the bounce diagram to plot $V(t)$ at a point midway along the length of the line from $t = 0$ to $t = 25$ ns.**

From the bounce diagram, we draw a vertical line at $z = 0.5$ m. We then sum up all the voltages that cross this vertical line to obtain the voltage sum after each bounce.



c **Find the steady-state voltage on the line.**

$$V_\infty = V_1^+ \frac{1 + \Gamma_L}{1 - \Gamma_L \Gamma_g} = (20) \frac{1 - 1/3}{1 - (-1/3)(1/3)} = 12$$

We find that the steady-state voltage on the line is the same as the voltage at the load in a voltage-dividing equivalent circuit.

$$V_\infty = \frac{V_g R_L}{R_g + R_L} = \frac{(60)(25)}{100 + 25} = 12$$